1) (10 Points) Consider the following linear system

$$\begin{cases} 3x_1 & - & 6x_2 & + & x_3 & + & 5x_4 & = & 5 \\ 2x_1 & - & 4x_2 & + & x_3 & + & 3x_4 & = & 4 \\ -x_1 & + & 2x_2 & - & 2x_3 & & & = & -5 \end{cases}.$$

   i) Find a matrix $A \in \mathbb{R}^{3 \times 4}$ and and a vector $b \in \mathbb{R}^3$, such that the solutions of the above linear system are given by the vectors $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4$ satisfying $Ax = b$.

   ii) Determine the row-reduced echelon forms of the matrices $(A \mid b)$ and A and calculate their ranks.

   iii) Find all the solutions to the linear system.

   iv) Determine all $x \in \mathbb{R}^4$ which satisfy $Ax = b$ and which are orthogonal to the vector $u = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$.

2) (8 Points) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$ and define the following three functions:

$$f_1 : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 2x_1 + 3x_2 \\ x_1 + (u \bullet u)x_3 \end{pmatrix},$$

$$f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \sin(x_1) + \cos(x_2),$$

$$f_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$x \longmapsto \begin{pmatrix} x \bullet x \\ 0 \\ u \bullet u \end{pmatrix}.$$

   i) Which of the above functions f_1, f_2, f_3 are linear maps? For each one that is linear, determine its matrix.

   ii) Is f_2 injective and/or surjective?

3) (8 Points)

   i) Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with

$$G\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}.$$

   Determine the matrix of G.

   ii) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with

$$F\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \quad F\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 6 \end{pmatrix}.$$

   Show that F is not injective.

4) (8 Points) We define the following linear map

$$H : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 - x_3 \\ x_1 + 2x_2 \\ x_2 + x_3 \end{pmatrix}.$$

   i) Calculate the image of H.

   ii) Decide if H is injective and/or surjective.

   iii) Find all vectors $x \in \mathbb{R}^3$ with $H(x) = 2x$.

<center>**After finishing this exam submit your solution as one pdf file
at NUCT at the "Midterm" assignment.**</center>